So, $\vec{n}$ is:

$$\vec{n} = \frac{\vec{r}_r \times \vec{r}_\theta}{\|\vec{r}_r \times \vec{r}_\theta\|} = \frac{\langle -2r^2\cos\theta, -2r^2\sin\theta, r\rangle}{r\sqrt{4r^2+1}}$$

◇

Lecture 21

Flux:

The concept of flux is measuring the rate at which something flows through a surface (e.g. air through a butterfly net).

$$d\vec{S} = \vec{n}\,dS = \frac{\vec{r}_u \times \vec{r}_v}{\cancel{\|\vec{r}_u \times \vec{r}_v\|}}\left(\cancel{\|\vec{r}_u \times \vec{r}_v\|}\,dA\right) = (\vec{r}_u \times \vec{r}_v)\,dA$$

Def: If $\vec{F}$ is a continuous vector field defined on a surface S which has orientation $\vec{n}$, then the flux of $\vec{F}$ across S is

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S (\vec{F}\cdot\vec{n})\,dS$$

$$\left[\begin{array}{l} S \text{ parametrized by } \vec{r}(u,v) \\ \vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} \end{array}\right] = \iint_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u \times \vec{r}_v)\,dA$$

Ex: Find the flux of $\vec{F} = \langle x, y, z \rangle$ across the helicoid parametrized by $\vec{r}(u,v) = \langle u\cos v, u\sin v, v \rangle$, $0 \le u \le 1$, $0 \le v \le \frac{\pi}{2}$ with upward orientation.

Sol: Start by finding the correct orientation.

$\vec{r}_u = \langle \cos v, \sin v, 0 \rangle$, $\vec{r}_v = \langle -u\sin v, u\cos v, 1 \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = \langle \sin v, -\cos v, u \rangle$$

Since the $\hat{k}$-component is positive, we have the right order. Since $\|\vec{r}_u \times \vec{r}_v\|$ cancells out in the flux integral there's no need to compute it.

$\vec{F}(\vec{r}(u,v)) = \langle u\cos v, u\sin v, v \rangle$

$\vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) = u\cos v \sin v - u\sin v \cos v + uv = uv$

So, the flux is:

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^{\pi/2} uv \, dv \, du = \int_0^1 \frac{1}{2} uv^2 \Big|_0^{\pi/2} du = \int_0^1 \frac{\pi^2}{8} u \, du$$

$$= \frac{\pi^2}{16}$$

◇

<u>Ex</u>: Compute $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = \text{curl}\,\vec{G}$, $\vec{G} = \langle -2yz, y, 3x \rangle$, and S is the piece of the paraboloid $z = 5-x^2-y^2$ above the plane $z=1$, with upward orientation.

<u>Sol</u>: Begin by parametrizing S. Using cylindrical, the paraboloid is $z = 5-r^2$, so:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \qquad\qquad 0 \le \theta \le 2\pi, \quad 0 \le r \le ? \\ z &= z = 5-r^2, \end{aligned}$$

To get the upper bound on r, we need to know the intersection of the paraboloid with $z=1$:

$$1 = z = 5-x^2-y^2 \iff x^2+y^2 = 4 \iff r = 2$$

So, a parametrization is

$$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 5-r^2 \rangle, \; 0 \le r \le 2, \; 0 \le \theta \le 2\pi$$

Now, $\vec{r}_r = \langle \cos\theta, \sin\theta, -2r \rangle$, $\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle 2r^2\cos\theta, 2r^2\sin\theta, r \rangle$$

Since the $\hat{k}$-component is positive, this was the correct order.

Now,

$$\vec{F} = \text{curl } \vec{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2yz & y & 3x \end{vmatrix} = \langle 0, -2y-3, 2z \rangle$$

So,

$$\vec{F}(\vec{r}(r,\theta)) = \langle 0, -2r\sin\theta - 3, 10 - 2r^2 \rangle$$

and

$$\vec{F}(\vec{r}(r,\theta)) \cdot (\vec{r}_r \times \vec{r}_\theta) = -4r^3\sin^2\theta - 6r^2\sin\theta + 10r - 2r^3$$

Finally,

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^2 \left(-4r^3\sin^2\theta - 6r^2\sin\theta + 10r - 2r^3\right) dr\, d\theta$$

$$= \int_0^{2\pi} \left(-r^4\sin^2\theta - 2r^3\sin\theta + 5r^2 - \tfrac{1}{2}r^4\right)\Big|_0^2 d\theta$$

$$= \int_0^{2\pi} \left(-16\sin^2\theta - 16\sin\theta + 12\right) d\theta \quad \left(\sin^2\theta = \frac{1-\cos 2\theta}{2}\right)$$

$$= \int_0^{2\pi} \left(8\cos 2\theta - 16\sin\theta + 4\right) d\theta$$

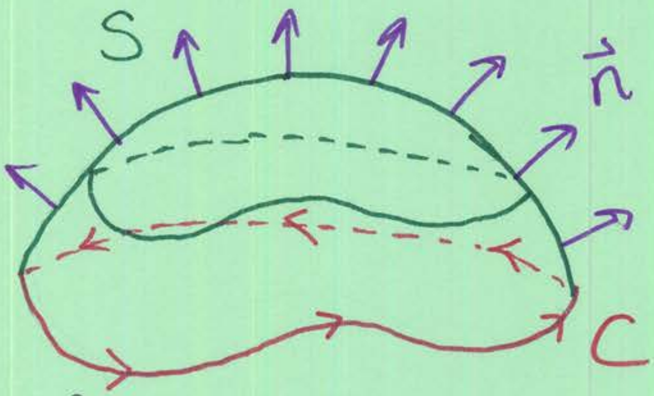
$$= \left(4\sin 2\theta + 16\cos\theta + 4\theta\right)\Big|_0^{2\pi} = 8\pi$$

◇

16.8 - Stokes' Theorem

Let S be a surface with boundary C and orientation $\vec{n}$:

ex:



We say that C has orientation consistent with S if while traversing C with your head in the direction of $\vec{n}$, the surface is on your left. In this sense, we say S induces an orientation on C, and C with this orientation is denoted ∂S.

Stokes' Theorem: Let S be an oriented, piecewise-smooth surface which is bounded by a simple, closed, piecewise-smooth curve C, and give C the orientation induced by S. Let $\vec{F}$ be a vector field on $\mathbb{R}^3$ which is C^1 in an open region containing S. Then

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\operatorname{curl}\vec{F})\cdot d\vec{S}$$

Let's revisit the example from earlier:

Alt. Sol.:

We can apply Stokes' theorem to this case. C is the circle $x^2+y^2=4$, $z=1$. The orientation we want on C is the counterclockwise one, when viewed from above. So, a parametrization of C is:

$$\vec{r}(t) = \langle 2\cos t, 2\sin t, 1\rangle, \ 0 \le t \le 2\pi.$$

By Stokes': $\iint_S \text{curl}\,\vec{G}\cdot d\vec{S} = \int_C \vec{G}\cdot d\vec{r}$,

so: $\vec{G}(\vec{r}(t)) = \langle -4\sin t, 2\sin t, 6\cos t\rangle$

$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 0\rangle$

$$\vec{G}(\vec{r}(t))\cdot\vec{r}\,'(t) = 8\sin^2 t + 4\sin t\cos t$$

$$\int_C \vec{G}\cdot d\vec{r} = \int_0^{2\pi} (8\sin^2 t + 4\sin t\cos t)\,dt$$

$$= \int_0^{2\pi} (4 - 4\cos 2t + 4\sin t\cos t)\,dt$$

$$= (4t - \sin t + 2\sin^2 t)\Big|_0^{2\pi} = 8\pi$$

$\Diamond$

Quicker!

<u>Ex</u>: Compute $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}=\langle xy, 2z, 3y\rangle$ and C is the curve of intersection between $x+z=5$ and $x^2+y^2=9$, oriented counterclockwise when viewed from above.

<u>Sol</u>: Parametrizing C wouldn't be too bad

$\vec{r}(t)=\langle 3\cos t, 3\sin t, 5-3\cos t\rangle$, $0\le t\le 2\pi$

but: $\vec{r}\,'(t)=\langle -3\sin t, 3\cos t, 3\sin t\rangle$

& $\vec{F}(\vec{r}(t))=\langle 9\cos t\sin t, 10-6\cos t, 9\sin t\rangle$

so:

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)=-27\cos t\sin^2 t+30\cos t-18\cos^2 t+27\sin^2 t\ \ldots$

awful...

But, we can still use Stokes' theorem! We need a surface with boundary on C. We may as well use the piece of the plane $x+z=5$ inside the cylinder $x^2+y^2=9$. So:

$\vec{r}(x,y)=\langle x, y, 5-x\rangle$, $D=\{x^2+y^2\le 9\}$

In order for S to induce the correct orientation on C, it needs the upward orientation. So:

$\vec{r}_x=\langle 1,0,-1\rangle$, $\vec{r}_y=\langle 0,1,0\rangle$, which has positive $\hat{k}$-component, so $\vec{r}_x\times\vec{r}_y$ is the correct order

$$\vec{r}_x\times\vec{r}_y=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 0 & -1\\ 0 & 1 & 0\end{vmatrix}=\langle 1,0,1\rangle,$$

$$\text{curl}\,\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xy & 2z & 3y\end{vmatrix}=\langle 1,0,-x\rangle,\quad (\text{curl}\,\vec{F})(\vec{r}(x,y))=\langle 1,0,-x\rangle.$$

Thus,

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_D \langle 1,0,-x\rangle\cdot\langle 1,0,1\rangle\, dA$$

$$= \iint_D (1-x)\,dA \overset{\substack{\text{switch to}\\ \text{polar}}}{=} \int_0^{2\pi}\int_0^3 (1-r\cos\theta)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 (r-r^2\cos\theta)\,dr\,d\theta = \int_0^{2\pi}\left(\frac{r^2}{2}-\frac{r^3}{3}\cos\theta\right)\Bigg|_0^3 d\theta$$

$$= \int_0^{2\pi}\left(\frac{9}{2}-9\cos\theta\right)d\theta = 9\pi.$$

$\diamond$

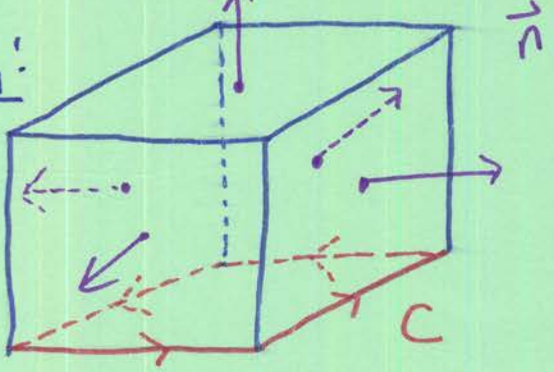
Let's close with one more example: a trick. Suppose S_1 & S_2 have the same boundary C, and they both induce the same orientation on C. If everything in question satisfies Stokes' theorem, then

$$\iint_{S_1} (\operatorname{curl}\vec{F})\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r} = \iint_{S_2} (\operatorname{curl}\vec{F})\cdot d\vec{S}.$$

I like to call this process "surface swapping".

Suppose we were asked to compute $\iint_{B_1} (\operatorname{curl}\vec{F})\cdot d\vec{S}$, where B_1 is the surface of the box $[0,1]\times[0,1]\times[0,1]$, with no bottom, where B_1 has the "outward" orientation.

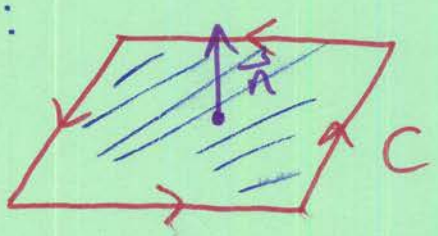
$\underline{B_1}$:



This would require computing $\underline{\underline{5}}$ surface integrals ... quite frustrating, and switching to an integral over the boundary C still requires 4 line integrals...

However, using surface swapping, we can replace B_1 by the bottom of the box, B_2, with upward orientation:

$\underline{B_2}$:



$$B_2 = [0,1] \times [0,1] \times \{0\}$$

Much easier!